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CUMULATED SOCIAL ROLES: THE DUALITY OF PERSONS AND THEIR ALGEBRAS *

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The study of social roles from the perspectives of individual actors, and the relation of graph homomorphisms to semigroup homomorphisms, have been the two most prominent topics to emerge from the recent resurgence of progress made on the algebraic analysis of social networks. Through our central construction, the cumulated person hierarchy, we present a framework for elaborating and extending these two lines of research. We focus on each actor in turn as ego, and we articulate what we believe to be the fundamental duality of persons and their algebras. We derive graph and semigroup homomorphisms for three algebras containing 81, 43, and 93 elements, respectively. Throughout, our discussion of theoretical issues is oriented toward an empirical application to the Padgett data set on conspiracy and faction in Renaissance Florence.

1. Introduction

Following the pioneering work of White (1963), Boyd (1969), Lorrain and White (1971), and Boorman and White (1976), there has been a resurgence of interest in the algebraic analysis of social networks. By "algebraic analysis" we refer to the study of how the various qualities of relationship interpenetrate in multiple network systems. Specifica-

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tion of how social roles interrelate is a core problem for any theory of social structure. [1]

Two research questions have been prominent in the recent resurgence of progress on algebraic analysis. One question concerns the relation between algebraic analysis and the more usual problem of how to partition actors into "cliques", "blocks", and so on. More specifically, the question is how to represent complex structures by simpler structures at two different levels, and the relation between these levels of simplification: (a) reduced-form representation of the actors and their relationships (as for example in blockmodel analysis), and (b) reduced-form representation of the algebra of relations (the interpenetration of qualities of relations, or "role interlock") across multiple networks of collective action. Major recent progress on the relationship between "graph homomorphisms" (topic (a) above) and "semigroup homomorphisms" (topic (b)) has been made by Bonacich (1982), Pattison (1982), White and Reitz (1982, 1983), and – most systematically – by Kim and Roush (1984).

A second prominent result of the recent progress on algebraic analysis has been a shift away from Boorman and White's focus on the global analysis of relational interlock (a focus which was also adopted in our own previous work; Breiger and Pattison 1978). Attention has shifted instead to the question of how the algebra impinges on the perspectives of *individual* actors who are enmeshed in multiple networks of relations. This thrust has led to the development of the concepts of "role-equivalence from the point of view of individuals" (Winship and Mandel 1983), "regular equivalence" (White and Reitz 1983) "local blockmodel algebras" (Wu 1983), "local role algebras" (Pattison 1980, 1986; Mandel 1983), and to the line of work critically assessed in Bonacich (1983).

In this paper we present a framework for elaborating these two types of concerns. Building on the recent work, we focus on each actor in turn as ego, and we exploit what we believe to be the fundamental duality between persons and their algebras. [2] Throughout, our treat-

1 See, for example, the general discussions of algebraic analysis and social theory in Boorman and White (1976), Berkowitz (1982), and Wellman (1983).

2 By "duality" we refer to the existence of structure at two different levels (for example, persons and relations, or points and lines) that are nonetheless unified in the sense that each level is composed of entities from the other (Breiger 1974; McPherson 1982; Fararo and Doreian 1984). Section 4 below elaborates this concept.

ment of ego algebras is informal, non-technical, and pitched toward the reader who is unfamiliar with algebraic analysis. We present the key concepts below, with reference to a data set (Padgett 1987) of great importance for the historical and sociological study of political organization and faction.

Sections 3–5 introduce our new developments, following an orientation to the data and to basic concepts (Section 2). Our data analysis is reported in Section 3 and 6. In Section 7 we mention the relevance of this analysis for *other* data sets we have analyzed. Section 8 provides a formal description of our new developments and suggests some theoretical extensions as well as data-analytic strategies. Section 9 begins with an explicit comparison of our approach with two others – regular equivalence (White and Reitz 1982), and local or "bundle" equivalence (Winship and Mandel 1983; White and Reitz 1983). Then, in extending the comparison, we present a new procedure for characterizing concrete social roles (Section 9.2).

2. Basic concepts and introduction to data on faction in Florence, 1426–1434

Our data for Sections 2–6 are drawn from John Padgett's significant study of party and faction in fifteenth-century Florence (Padgett 1987). Intricate networks of conspiracy and connection evolved over several centuries among the ruling oligarchical families who are the subjects of Padgett's analysis. Politics in Florence was the conflict of wealthy families and factions for control of the government. Surely the most notable of the conspiracies came to fruition on 9 September 1433, when opponents of the Medici family gained sufficient control of the council of government to order the entire family, as well as many of its chief supporters, exiled from the Florentine Republic. [3] Revenge was to be gained precisely twelve months later, when the Medicis and their supporters returned from their exile in Venice to permanently banish or otherwise punish their enemies and to begin their lengthy reign. This short gloss on the historians' accounts does them an injustice; standard sources include Brucker (1969), Kent (1978), Najemy (1982), Rubinstein (1966), and Waley (1969).

Padgett meticulously coded the financial and marriage relations

[3] Several members of the family were excepted from this punishment; see Kent (1978: 295).

among 116 of the leading Florentine families, using the most detailed extant narrative account of the period (Kent 1978) as his primary source. This example of finely elaborated analysis was made possible by the extraordinarily rich archives of tax records (including the tax returns that individual families filed after an income tax was instituted in 1427), business records, electoral lists, personal diaries, and files of personal correspondence housed in Florence and available to historians. Padgett's "financial" relations include the granting of credit, loans, and joint business partnerships. These are business relations with overtones of patron–client connections.

Two features of Padgett's coding scheme are most relevant for our present purposes. First, he takes as his unit of analysis the family, not the individual. Historians would no doubt disagree among themselves in their evaluation of this coding decision, but in this respect Padgett is consistent with Kent (1978: 194), whose discussion of the question concludes by noting that, with some qualification, the evidence strongly suggests that families were the naturally-existing interest groups, that they actually served as the basis for fractions, and that they tended to unify internally in the face of political crisis. Second, Padgett coded all ties as symmetric. This is unfortunate for our purposes, since it does

Table 1
Business and marriage relations among 16 families in fifteenth-century Florence (data of John Padgett) [a]

	Business	Marriage
1. Acciaiuoli	....	........X.......
2. Albizzi	....	.....XX.X.......
3. Barbadori	....XX..X.X.....	....X...X.......
4. Bischeri	......XX..X.....	......X...X...X.
5. Castellani	..X....X..X.....	..X.......X...X.
6. Ginori	..X.....X.......	.X..............
7. Guadagni	...X...X........	.X.X...X.......X
8. Lamberteschi	...XX.X...X.....	......X.........
9. Medici	..X..X...X...X.X	XXX.........XX.X
10. Pazzi	........X.......	.............X..
11. Peruzzi	..XXX..X........	...XX.........X.
12. Pucci	....	....
13. Ridolfi	....	........X.....XX
14. Salviati	........X.......	........XX......
15. Strozzi	....	...XX.....X.X...
16. Tornabuoni	........X.......	......X.X...X...

[a] Families listed alphabetically. All data coded as symmetric.

not allow us to take account of the social asymmetry in relationships such as the granting of loans. With Padgett, we plan to revise the coding scheme and to provide a substantive analysis of the full network data generated by these 116 families. Our purposes here is to illustrate an applied analytical framework.

Shown in Table 1 is the small portion of Padgett's network data that we will use. Our sample of 16 of Padgett's 116 families was chosen deliberately, as an aid to our own intuition, to represent some of the families whose support of, or opposition to, the Medicis has been clearly established. Rows and columns are listed identically in the alphabetical order of the families named in the left margin. Financial and marriage ties are reported separately, with an "X" indicating the presence of a tie. Business ties (primarily loans) are designated matrix $\boldsymbol{L}$, and marriage ties matrix $\boldsymbol{M}$.

2.1. The data semigroup

In this and the following subsection we review some basic algebraic concepts, while also applying them to the Florentine data. We first consider all compound relations generated by the $\boldsymbol{L}$ and $\boldsymbol{M}$ networks of Table 1. For example, Barbadori has a financial tie to Ginori, and Ginori has one to Medici (entries [3,6] and [6,9] of the $\boldsymbol{L}$ matrix), so Medici is the partner of a partner of Barbadori. A direct financial tie also connects Barbadori and Medici (entry [3,9] of the $\boldsymbol{L}$ matrix) so, with respect to these three families, the "partner of a partner" (a compound tie) is a "partner" (a direct tie).

This equation (namely, $\boldsymbol{L}^2 = \boldsymbol{L}$) is not true of the families in general, as can be seen by tracing out all ties of length two in the $\boldsymbol{L}$ matrix, in just the manner illustrated above, or – equivalently – by computing the Boolean matrix product $\boldsymbol{L}^2$ and comparing it with matrix $\boldsymbol{L}$. Since the $\boldsymbol{L}$ and $\boldsymbol{L}^2$ matrices do not coincide, we therefore say that the abstract role relations ("partner of a partner" and "partner") indexed by these matrices are "distinct". [4]

[4] We thus employ a version of the "Axiom of Quality" of Boorman and White (1976: 1393): two types of abstract role relationship are identified if and only if their associated matrices coincide. Our application is somewhat unusual, though, in that we apply this axiom to the full network data, rather than to aggregated blockmodel "images" derived from the data; see below. It is troubling that equations such as $\boldsymbol{L}^2 = \boldsymbol{L}$ are true for *some* triads (as illustrated above in the text) and yet rejected as a characterization of "global" role relations. Such discomfort has provided one motivation for the development of "local" role analysis. A particularly clear treatment of the distinction between "local" and "global" levels of role analysis is provided by Mandel (1983).

We are concerned with the entire system of relations among these families. We therefore continue to generate compound relations implied by the L and M matrices, until no new distinct compounds are generated. As it happens, there are 79 distinct compounds. All equalities among these 81 matrices (the generators of Table 1, and the compounds) are shown in Table 2. As an example of constructing (and reading) Table 2, we begin with two rows and two columns (both sides labeled 1 and 2 for L and M) and no entries. The first entry reports that the product of matrix $1 = L$ with itself is equal to neither of the generators (neither L nor M). This compound is thus given the name of the next ascending integer ($3 = L^2$). The integer name is entered in the (1, 1) cell of Table 2, and a new row (the third) is added to the table. This process continues until no new products are discovered. As a system for portraying systematically all interrelations of the L and M networks, Table 2 is termed the "semigroup multiplication table generated by L and M". Crucially, Table 2 is closed under the operation of the composition of relations (which is to say that the product of any two of the 81 relations indexed by Table 2 is itself one of these 81 relations).

Readers familiar with the literature on blockmodels will observe that the procedure for generating the Table 2 semigroup is identical to the techniques presented by Boorman and White (1976), but with one crucial exception: We have not begun with a blockmodel. We generate Table 2 from the full multiple network data of Table 1. To contrast the Table 2 semigroup with the "blockmodel semigroups" analyzed by Boorman and White, we refer to Table 2 as the "data semigroup". As in Boorman in White's formulation (which also serves as an instructive introduction to this route to algebraic analysis; see also Berkowitz 1982: 50–54, 91–101), the semigroup is a system; it is an algebra closed under the operation of forming compounds.

Because the data semigroup provides "a complete statement of the role structure in the given data" (Boorman and White 1976: 1397), analysts have sought simplifications of this algebra that preserve its properties of relational composition. A "homomorphism" of the semigroup consists of a partition of its elements into classes such that the composition of relations among these classes is consistent with the composition of relations in the underlying table (in a manner that is made precise in Boorman and White 1976: 1418, and in Bonacich 1983: 175).

Table 2
The 81-element semigroup multiplication table for the data of Table 1 [a]

	L	M		L	M
$1 = L$	3	4	41	59	61
$2 = M$	5	6	42	59	60
$3 = L^2$	7	8	43	59	62
$4 = LM$	9	10	44	63	64
$5 = ML$	11	12	45	59	60
$6 = M^2$	13	14	46	59	65
$7 = L^3$	15	16	47	66	67
$8 = L^2M$	17	18	48	66	68
$9 = LML$	19	20	49	66	69
10	21	22	50	66	70
11	23	24	51	66	71
12	25	26	52	72	73
13	27	28	53	66	74
14	29	30	54	66	68
15	31	32	55	31	56
16	31	33	56	31	56
17	31	32	57	31	56
18	31	34	58	31	56
19	31	35	59	59	75
20	36	37	60	59	76
21	31	38	61	59	76
22	31	39	62	59	76
23	40	41	63	59	77
24	42	43	64	59	78
25	40	44	65	59	76
26	45	46	66	66	79
27	47	48	67	66	68
28	49	50	68	66	68
29	51	52	69	66	68
30	53	54	70	66	68
31	31	55	71	66	68
32	31	56	72	66	80
33	31	56	73	66	81
34	31	56	74	66	68
35	31	56	75	59	76
36	31	57	76	59	76
37	31	58	77	59	76
38	31	56	78	59	76
39	31	56	79	66	68
40	59	60	80	66	68
			81	66	68

[a] Although the table is of size 81×81, only the first two columns are shown here. Other entries may be found by associativity. For example: the product $2 \times 3 = 2 \times (1 \times 1) = (2 \times 1) \times 1 = 5 \times 1 = 11$.

2.2. Local structure

The key analytical problem about semigroup homomorphisms is posed sharply by Bonacich (1983: 173), who writes: "While it is clear what homomorphisms are in terms of semigroups, it is unclear what they mean in terms of the data matrices that generate the semigroups. Homomorphisms are algebraic simplifications, but do they imply corresponding simplifications of the data?"

A major avenue of attack on this problem of the interpretability of a semigroup has been to recast the problem from the point of view of each individual actor as ego (as in the previously cited work of Mandel,

Table 3
RELE for Ridolfi (= family 13) taken as ego, and the associated right semigroup multiplication table

A: RELE (matrix A; rows index elements in Table 2, and columns index families in Table 1)

```
                 1111111
         1234567890123456
1 L      ................
2 M      ........X.....XX
5 ML     ..X..X..XX...X.X
6 MM     XXXXX.X.X.X.XX.X
11 MLL   ..X.XX..XXX..X.X
12 MLM   XXX.X.X.XX..XX.X
13 MML   ..XXXXXXXXX..X.X
14 MMM   XXXXXXXXXXX.XXXX
23 MLLL  ..XXXX.XXXX..X.X
24 MLLM  XXXXX.X.XXX.XXXX
28 MMLM  XXXXX.XXXXX.XXXX
```

B. Ridolfi's right semigroup multiplication table (reports Boolean products of *rows* L and M of the RELE with all *matrices* of Table 2; by construction, all such products are *themselves* rows of the RELE)

	$1 = L$	$2 = M$
$1 = L$	1	1
$2 = M$	5	6
$5 = ML$	11	12
$6 = MM$	13	14
$11 = MLL$	23	24
$12 = MLM$	23	14
$13 = MML$	13	28
$14 = MMM$	13	14
$23 = MLLL$	13	24
$24 = MLLM$	13	14
$28 = MMLM$	13	14

White and Reitz, Wu, Winship and Mandel, Bonacich, and Pattison). Our approach to this type of recasting is illustrated in Table 3.

Let us look at the problem of role interlock through the eyes of Ridolfi (family 13 in Table 1). In particular, let us consider the ties that this actor *sends* to the others. [5] This family does not "see" the full structure of relations that is accessible to the omniscient data analyst, but they are manifestly aware of their own ties. Ridolfi's rows in the L and M matrices are reproduced as the first two rows of panel A of Table 3. We may now consider compound relations formed by composing these *rows* with the L and M matrices or any of their compounds. For example, the Boolean product of Ridolfi's row in M (an array of size 1×16) with the matrix L^2 (an array of size 16×16) gives us this ego's row in the matrix MLL. Ego's row in MLL is reported as the fifth row in panel A. It is indexed by the number "11" since MLL is the eleventh semigroup element listed in Table 2. (This finding also gives us the equation "$5 \times 1 = 11$" in panel B of Table 3; see below.) The matrix in panel A of Table 3 is closed under the operation of composing rows of L or M with any product, of any length, involving the L or M matrices.

We term the rows of panel A of Table 3 the "row elements" (or RELE, for short) of Ridolfi as ego. The set of RELE is quite similar to the "relation plane" of Winship and Mandel (1983) and Mandel (1983), on whose work we build, although there are differences as well, which we set out in a footnote. [6]

5 Since these data are symmetric, the analysis of ties *received* by any ego would be identical. In general (for non-symmetric data), however, this discussion needs to be extended. We provide the extension below, in Sections 7 and 8.

6 If two rows of the "relation plane" are identical, Winship and Mandel decline to equate them (that is, they decline to impose the "Axiom of Quality", see footnote 4). In their formulation, the (theoretical) relation plane thus has an infinite number of rows. But in practice, these authors study the "truncated relation plane", which consists of ego's rows in all products (the original networks plus compounds of these relations) of less than a certain length. For example, these authors might study the relations L, M, L^2, LM, ML, and M^2 (products of length 1 or 2). Ego's "truncated relation plane" would then have six rows, comprising his rows in each of these matrices. By this or some similar analytical restriction of scope, the truncated relation planes of any *two* individuals are made conformable (e.g. both planes consist of six rows indexing the same relations in the same order, and of n columns indexing the n actors in the same order). Conformability is purchased, however, at the expense of loss of concern for the data semigroup as a system. Notice that each of the infinite number of rows in the Winship–Mandel (theoretical) relation plane is identical to exactly one row in the construction we propose, the RELE, whereas each of the infinite number of rows in the (theoretical) relation plane is not necessarily identical to any row in the truncated relation plane of Winship and Mandel.

Just as we formed the data semigroup from the matrices in Table 1, we may form ego's "right semigroup" multiplication table from the RELE in panel A of Table 3. Ridolfi's right table, for example, is reported directly below the RELE, in panel B. A useful result is that the right semigroup table of *each* ego is a right homomorphic image of the full data semigroup (Mandel 1978; Wu 1983; and see Section 3 below).

The advantage of such "local algebras" are sketched by Mandel and by Wu. These algebras describe the pattern of relational interlock with respect to *particular* actors. And because they induce simplifications of the role structure from local configurations of observed network bonds, it is clear what the simplifications mean substantively (Wu 1983: 291).

3. Aggregation of ego algebras

An outstanding problem in the analysis of ego algebras is the problem of aggregation *across* individuals. [7] Moreover, this aggregation problem has dual facets: observed social connections (e.g. Table 1 above) and the interlock of abstract qualities of relationship (Table 2).

On the algebraic side two approaches can be taken, which correspond to standpoints clarified in the extensive discussion in the social networks literature about "common" versus "joint" structure (Bonacich 1980; Bonacich and McConaghy 1979; McConaghy, 1981a, 1981b; Boorman and Arabie 1980; Pattison 1981; Wu 1983; Mandel 1983). Both types of aggregation can be defined for ego algebras.

First consider the equations that are true in *any* ego's algebra (such as Ridolfi's: panel B of Table 3). The intersection or "meet" of *all* the egos' right algebras is exactly the *full* data semigroup (of size 81×81 in our example) previously reported in Table 2 (see Wu 1983, for further discussion). This capturing of the full distinctiveness of all individuals by the full data algebra provides an important perspective on what the full algebra "means": it consists entirely and only of the structure of

7 In this respect we follow Robert Merton's assertion that "the notion of the role-set at once leads to the inference that social structures confront men with the task of *articulating* the components of countless role-sets" (Merton 1968: 42, emphasis added). For an argument that role structure is not simply a very long list of equations that happen to hold in a population (e.g. Table 2 above), but rather an *aggregation* of these equations, see Pattison (1981).

the ensemble of the individuals, without (so to speak) sacrificing the individuality of any of them.

Second, consider the algebraic simplification that would need to take place if we were to *combine* any two egos: for example, if we were to assign them to a jointly held "position" in the social structure. This simplification may be given by the "join" of the two algebras, which encompasses the inclusions in their union (Mandel 1983: 381).

Our goal is to discover reduced-form images ("graph homomorphisms") of the network connections that simultaneously lead to simplifications ("semigroup homomorphisms") of the full data algebra. In brief: we are concerned with the implications of *combining* individuals into hypothesized structural "positions" or "blocks". We therefore (unlike Wu 1983) have a natural interest in the "joint" structure of nay two right algebras.

An algorithm for computing the joint homomorphic reduction of two *full* algebras is the JNTHOM procedure of Harrison White (Boorman and White 1976: 1421). It is straightforward to rewrite this algorithm to compute the joint *right* homomorphism of two *ego* algebras, and we have done so. [8]

We are now in a somewhat novel position. Boorman and White investigated the distances between the algebras of the various data sets they studied (see their "Global Geometry for Types of Role Structures", 1976: 1426–1441). However, if only because the concepts associated with "local" structure had not yet been put forward, they did not extend their "geometry" to search for blockings (i.e. assignments to structural positions) of individuals within the *same* data set. Given our "ego-oriented" version of the JNTHOM algorithm, however, we are now in a position to use the data algebra to produce an aggregation of *actors* which maximally *conforms* to the data algebra. [9]

Toward this end, we computed the joint homomorphic reduction of each pair of "right" ego algebras for the 16 Florentine families. Using the "delta" measure of Arabie and Boorman (1973), which is in fact a semimetric, we then computed the *distance* between the right semigroups of each pair of families. (This is the same measure employed

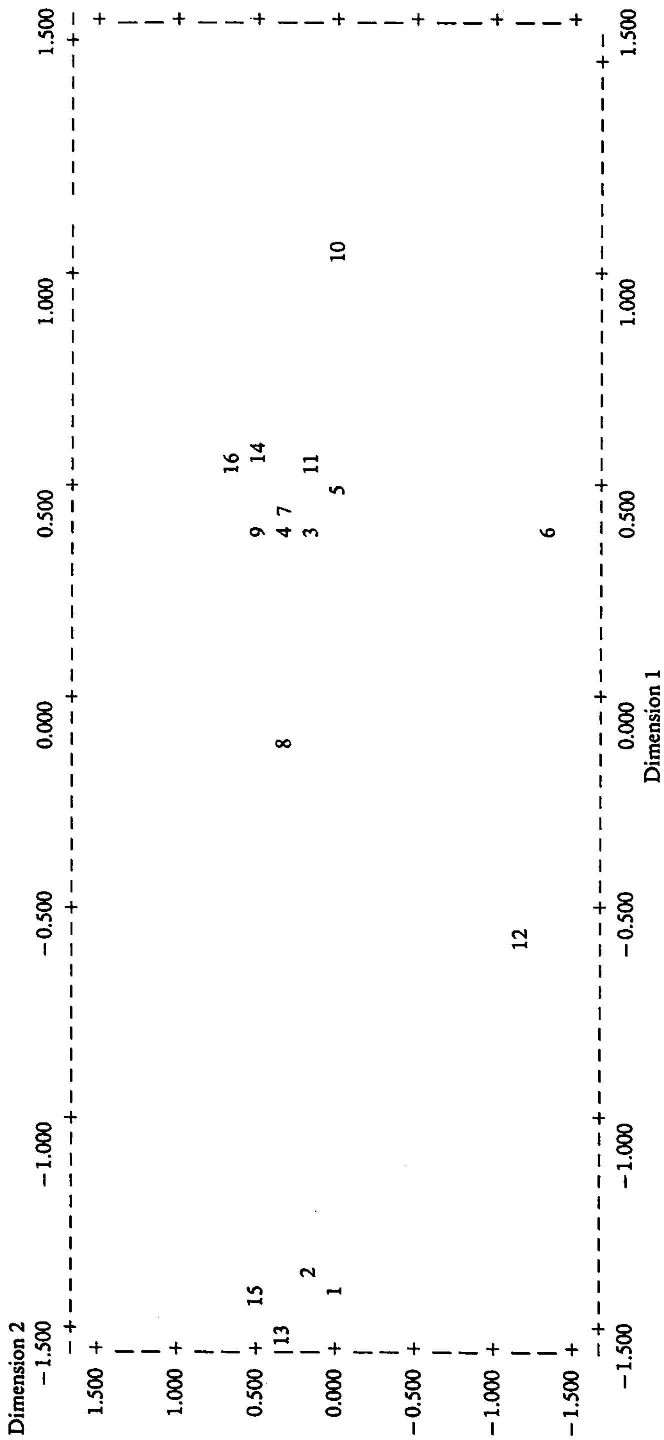
[8] The resulting algorithm, named RJNTHOM, written in the APL language, is available on request from the authors. It will run on any personal computer that supports "APL*Plus/PC", marketed by STSC, Inc., Rockville, MD.

[9] Existing models of local structure (those referenced in Section 1 of this paper) have the same goal. We provide explicit comparisons below, in Section 9.

Table 4
Distances between ego algebras
A. Distances (×100 and rounded) between each pair of right semigroup multiplication tables. (This is Boorman and White's 1976: 1423 measure of the distance between two semigroups, but applied here to right semigroup algebras)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	6	79	100	87	118	81	61	110	135	106	90	33	134	17	134
2	6	0	69	106	77	109	105	56	119	127	97	88	36	143	38	126
3	79	69	0	17	20	59	24	40	19	41	15	89	98	24	98	29
4	100	106	17	0	32	74	21	37	5	58	18	90	103	27	84	35
5	87	77	20	32	0	71	16	50	39	45	8	89	120	31	120	37
6	118	109	59	74	71	0	86	98	80	138	81	91	148	93	148	97
7	81	105	24	21	16	86	0	47	11	52	18	94	97	22	83	23
8	61	56	40	37	50	98	47	0	40	63	57	94	99	71	85	69
9	110	119	19	5	39	80	11	40	0	71	36	89	100	26	80	25
10	135	127	41	58	45	138	52	63	71	0	28	92	150	74	150	74
11	106	97	15	18	8	81	18	57	36	28	0	88	96	34	96	30
12	90	88	89	90	89	91	94	94	89	92	88	0	82	92	82	92
13	33	36	98	103	120	148	97	99	100	150	96	82	0	107	18	107
14	134	143	24	27	31	93	22	71	26	74	34	92	107	0	107	21
15	17	38	98	84	120	148	83	85	80	150	96	82	18	107	0	107
16	134	126	29	35	37	97	23	69	25	74	30	92	107	21	107	0

B. Multidimensional scaling of the distances in panel A



toward different ends in Boorman and White 1976: 1426–33.) These distances are reported in the top panel of Table 4. The bottom panel of Table 4 displays a multidimensional scaling of the distances (Kruskal's stress formula 1 = 0.123).

The thrust of our argument is that individual egos who share many features of their algebras (i.e. their right semigroup multiplication tables, an example of which is provided in panel B of Table 3) are likely candidates for aggregation into joint structural "positions" or "blocks" which will *also* lead to a direct simplification of the full data algebra.

Inspection of panel B of Table 4 provides some clear speculation about such an aggregation of actors. Families 1, 2, 13, and 15 seem clearly similar in their algebras, and collectively dissimilar from families 3, 4, 5, 7, 8, 9, 10, 11, 14, and 16, though 8 appears somewhat marginal from this second cluster. Two individual families, 6 and 12, seem far apart from all others, including each other. [10]

We could now move directly to the aggregation of actors into a blockmodel, *on the basis* of the similarity in ego algebras suggested by panel B of Table 4. We could then *check* whether this aggregation of individuals in fact produced a semigroup homomorphism of the Table 2 algebra. But we would very much prefer a more direct approach to aggregation, with the goal of reducing or eliminating the need to check on the algebra "side". It is to such an approach that we now turn.

4. The duality of persons and relations

We build fundamentally on the fact that the RELE of Table 3 (panel A) encode a dual structure. Let us examine separately the rows and columns of this matrix.

The third row (labeled *ML*) of the RELE reports that Ridolfi is married to a business partner of families 3, 6, 9, 10, 14, and 16. The fifth row (labeled *MLL*) reports that this ego is married to a business partner of a business partner of families 3, 5, 6, 9, 10, 11, 14, and 16. Notice that the second set of role partners (those connected to ego by

[10] The four-block CONCOR split of the distances in panel A of Table 4 is ⟨1, 2, 13, 15⟩, ⟨12⟩, ⟨3, 4, 5, 7, 8, 9, 10, 11, 14, 16⟩, ⟨6⟩. The next partition of the third-listed group yields ⟨8⟩ versus the rest.

the relation MLL) includes the first set (those connected to ego by the ML relation). Following Mandel (1983), we therefore say that relation ML is *included within* relation MLL *with respect to ego*. The set of all such inclusions forms a partial order, which we term ego's "relation hierarchy". Ridolfi's relation hierarchy is reported in panel A of Figure 1, both in matrix and in graph form. [11]

Now consider *columns* of the RELE in panel A of Table 3. The first column reports that Ridolfi sends ties of M^2, MLM, M^3, ML^2M, and M^2LM to family 1. The third column reports that Ridolfi send ties of ML, M^2, ML^2, MLM, M^2L, M^3, ML^3, ML^2M, and M^2LM to family 3. Notice that the first set of relations is included in the second set. In the language of White and Reitz (1983: 206–208), the "bundle" of relations for the tie (13, 3) includes the "bundle" of relations associated with the tie (13, 1). Winship and Mandel would say that the "role relation" (13, 3) includes the "role relation" (13, 1). [12]

What does this mean? It means that *every relation, of whatever type, linking ego and family 1 is also a relation, of the same type, between ego and family 3.* In this sense, family 1 is *included within* family 3, with respect to ego (= family 13). To appreciate the generality of this finding, consider its formalization. Let R^* be *any* relation constructed as one of the infinite number of products, of any length, that might conceivably be formed from the L and M matrices of Table 1. That is: R^* is any member of the class $R \times R \times R \times \ldots$, where any element in this product may be *either* L *or* M and there may be any number of elements. For any such product we choose to consider, we know that

the tie iR^*j implies the tie iR^*k

for $j =$ family 1, $k =$ family 3, $i =$ ego = Ridolfi = family 13, and for any conceivable relation R^*. We know that this is true because, first, family 13's RELE includes precisely all the distinct 13th rows of all the infinite number of matrices of the form R^*, and, second, because we found by inspection of the RELE that the third entry of any row includes the first entry. We refer to this property as the "relational inclusion" of j in k, with respect to ego i.

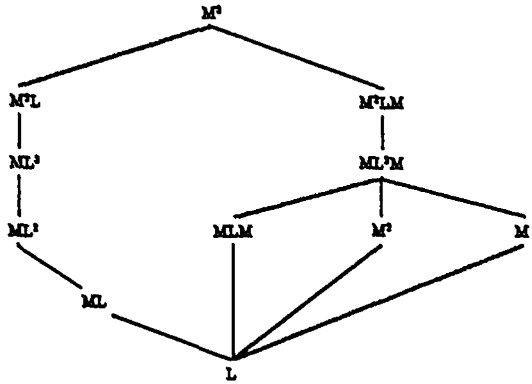
We refer to the entirety of ego's inclusions such as this one as ego's "person hierarchy", which is a partial order of social actors with

[11] A very similar idea is the "containment set" of Mandel (1983: 380). But see footnote 6 above.
[12] This statement is indicative but not completely correct; see again footnote 6.

$W = f(AA^{\mathrm{T}})$

```
                1111222
            12561234348

 1 L        XXXXXXXXXXX
 2 M        .X.....X.XX
 5 ML       ..X.X.XXX..
 6 MM       ...X...X.XX
11 MLL      ....X.XXX..
12 MLM      .....X.X.XX
13 MML      ......XX...
14 MMM      .......X...
23 MLLL     ......XXX..
24 MLLM     .......X.XX
28 MMLM     .......X..X
```

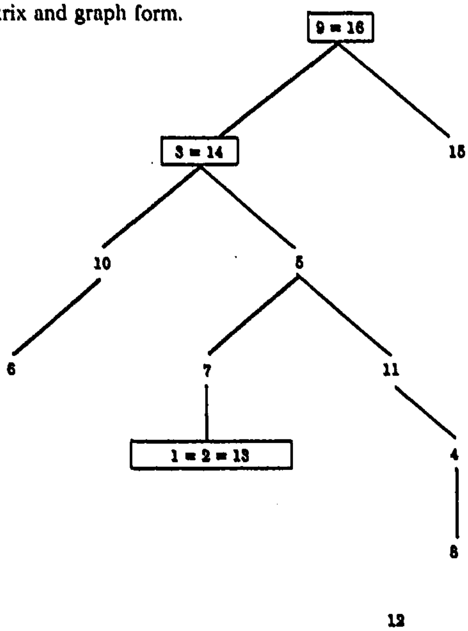


(a) Relation hierarchy (W) for Ridolfi, in matrix and graph form.

$P = f(A^{\mathrm{T}}A)$

```
                            1111111
                   1234567890123456

 1. Acciaiuoli     XXX.X.X.X...XX.X
 2. Albizzi        XXX.X.X.X...XX.X
 3. Barbadori      ..X.....X....X.X
 4. Bischeri       ..XXX...X.X..X.X
 5. Castellani     ..X.X...X....X.X
 6. Ginori         ..X..X..XX...X.X
 7. Guadagni       ..X.X.X.X....X.X
 8. Lamberteschi   ..XXX..XXXX..X.X
 9. Medici         ........X......X
10. Pazzi          ..X.....XX...X.X
11. Peruzzi        ..X.X...X.X..X.X
12. Pucci          ................
13. Ridolfi        XXX.X.X.X...XX.X
14. Salviati       ..X.....X....X.X
15. Strozzi        ........X.....XX
16. Tornabuoni     ........X......X
```



(b) Person hierarchy (P) for Ridolfi, in matrix and graph form.

Figure 1. The W and P matrices for Ridolfi as ego (A refers to the RELE of Table 3).

respect to ego. Ridolfi's person hierarchy is reported in panel B of Figure 1. Notice that certain sets of families (for example, family 1 and family 2) are equated in Ridolfi's person hierarchy, since each member of the set is included in each other member. We follow Mandel (1978) in terming such equivalence "partial structural equivalence (PSE) with respect to ego". Formally, two actors j and k are *PSE with respect to the ith ego* if and only if

(i) the tie iR^*j implies the tie iR^*k *and* the tie iR^*k implies the tie iR^*j, for any and all conceivable relations R^*, and
(ii) there is at least one relation R^* for which iR^*j.

The first axiomatic principle of our formulation of ego algebras is that ego's relation hierarchy (call it W) and ego's person hierarchy (call it P) are dual to each other, in the precise sense that both are simply derivative of a common underlying structure, the RELE. Denoting the latter matrix as A, the derivation is as follows:

$$W(i, j) = \begin{cases} 1 & \text{iff } A(i, k) \leqslant A(j, k) \text{ for all } k = 1, 2, \ldots, n, \\ 0 & \text{otherwise;} \end{cases}$$

$$P(i, j) = \begin{cases} 1 & \text{iff } A(k, i) \leqslant A(k, j) \text{ for all } k = 1, 2, \ldots, n, \\ & \text{and there is some k for which } A(k, i) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the case where the network is completely connected (i.e. where the union of the generator and compound ties R^* is a matrix filled with 1's), we may write

$$W = f(AA^{\mathrm{T}}), \qquad P = f(A^{\mathrm{T}}A),$$

where the composition is ordinary matrix multiplication, superscript T denotes transposition, and f is the operation of binarizing the product according to whether or not all entries in a given row are identical to that row's diagonal entry (compare Breiger 1974). [13]

We note from the definition of P that a person who has no

[13] Thus, with respect to W, for example, f is the operation of converting an integer-valued matrix $M = AA^{\mathrm{T}}$ to a binary matrix of inclusions, W, by imposing that $W(i, j) = 1$ if and only if $M(i, j) = M(i, i)$. What this means is that $W(i, j) = 1$ if and only if row i of A is contained in row j of A. A similar argument (but with respect to columns of A) applies to P. A more intuitive definition of W and P have already been provided in the text.

connections to any other person has a zero row and column in P. Indeed, P expresses inclusions only among persons in connected components of the network considered as a multigraph. (The non-symmetric case is discussed below in Sections 7 and 8.) If a network happens to have two or more connected components, then P expresses inclusions among persons within each component and there are no inclusions between persons in different components. Actors not belonging to *any* component, such as family 12 in the Florentine network, are simply not referred to by P: there is no relational evidence with which to examine the local role of family 12.

Defining P in this way emphasizes the *concrete* nature of individuals' roles. $P(i, j) = 1$ only if any relation between i and k is always accompanied by the same type of relation between j and k, *and* if there is at least one person k for whom this is so. The definition also has the important property that the P matrix for each component of a network is independent of the number and nature of the components in the network. The analysis is directed to take place *within* components of the network: that is, among sets of network members who are tied, possibly indirectly, to one another.

The definition of "partial structural equivalence" is similar, but not identical, to the "role equivalence" of Winship and Mandel (1983) and to the "regular equivalence" of White and Reitz (1983). One crucial difference, however, is that these latter forms of equivalence are more abstract than is PSE, in that they define equivalence in a way that is not constrained by specific connections among individuals. By way of contrast, individuals who are PSE with respect to an ego have the same kinds of relations to *precisely the same* set of individuals. The second difference is that, in practice, role and regular equivalence direct the analyst's attention to the degree of similarity of the entire RELE of a pair of individuals. (Our analysis in Section 3 also compared the "degree" of similarity of pairs of individuals, although it was their algebras, rather than their RELE, that provided the focus of comparison; see also Mandel 1978 for a related method of analysis.) The "overall" structure is then delineated from these measures on pairs of actors. In contrast, PSE (or, more generaly, relational inclusion) focuses the analyst's attention on whether or not two actors are identical from the standpoint of a *single* actor as ego. The "overall" structure is delineated by aggregating these Boolean "measures" (namely, inclusions) across all the actors, in a manner that we are about to describe.

5. The cumulation of local structure

From Ridolfi's partial order of individuals (shown graphically in Figure 1), we see at once that, from this ego's perspective, families 9 and 16 are PSE, as are families 3 and 14, and as are families 1, 2, and 13 (the latter being ego itself). The meaning of this "equivalence" was developed above: for example, Ridolfi encounters family 9 if and only if he encounters family 16, and this statement is true across any and all of the infinite number of relations implied by $\boldsymbol{L}$ and $\boldsymbol{M}$.

If we now consider some other family as ego, this second actor will in general have *different* sets of equivalent people and, more generally, a different partial order of person-inclusions; that is to say, the $\boldsymbol{P}$ matrices of any two actors will not in general coincide.

Now consider *all* social actors simultaneously as ego. *What would it take to make all of their inclusions true simultaneously*? Our answer to this question is: concrete social structure. More specifically, we view concrete social structure as the constraints acting upon a relational system which enforce all the inclusions and equivalences that impinge upon all individuals, each acting as an "autonomous" ego, to hold simultaneously.

Our particular models, or reduced-form images, of social structure derive from imposing the assumption that all individuals' inclusions operate simultaneously. For such simultaneity to operate – that is, to maintain as a collective social "act" the roles played by all the actors – *additional* inclusions (over and above the sum of the inclusions that hold for each individual as ego) must be imposed. We seek a reduced-form image of social structure in which the aggregated sets of "equivalent" social actors enforce more than the sum of the equivalences of the actors each taken as ego.

These considerations lead us to examine the union of the person hierarchies (the $\boldsymbol{P}$ matrices), taken across all 16 individuals in turn as ego. In the union, all individuals' person-inclusions simultaneously hold, although from the point of view of any *particular* ego additional inclusions have (in general) been added.

But we do not stop here. We contend that there is a *collective* partial order of person-inclusions. This implies transitivity at the aggregate level: if actor a is included in b and b is included in c, then a must be included in c. To the union of all $\boldsymbol{P}$ matrices we therefore add the additional inclusions necessary for transitivity to hold. (In more techni-

```
                    11      1111 1
                    1235345789014662

 1. Acciaiuoli      XXXXXXXXXXXXXX..
 2. Albizzi         XXXXXXXXXXXXXX..
13. Ridolfi         XXXXXXXXXXXXXX..
15. Strozzi         XXXXXXXXXXXXXX..
 3. Barbadori       ....XXXXXXXXXX..
 4. Bischeri        ....XXXXXXXXXX..
 5. Castellani      ....XXXXXXXXXX..
 7. Guadagni        ....XXXXXXXXXX..
 8. Lamberteschi    ....XXXXXXXXXX..
 9. Medici          ....XXXXXXXXXX..
10. Pazzi           ....XXXXXXXXXX..
11. Peruzzi         ....XXXXXXXXXX..
14. Salviati        ....XXXXXXXXXX..
16. Tornabuoni      ....XXXXXXXXXX..
 6. Ginori          ....XXXXXXXXXXX.
12. Pucci           ................
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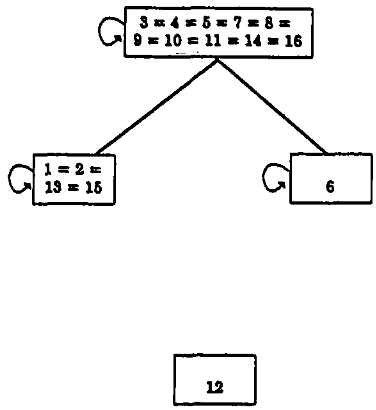


Figure 2. The cumulated person hierarchy, in matrix and graph forms.

cal language, we compute the transitive closure of the union of all individuals' $\boldsymbol{P}$ matrices.) We term the resulting partial order (and its matrix representation) the "cumulated person hierarchy".

For the data of Table 1, we report this aggregated hierarchy in both matrix and graph form in Figure 2.

6. Analysis of the Table 1 networks

Interpretation of the cumulated person hierarchy is straightforward. Interpretive power comes from the *zeros* in this matrix. A zero in the (j, k)th cell means that *for no ego* is it true that actor j is relationally contained in actor k. In particular: for no social actor as ego is it true that actors j and k are relationally equivalent, as this concept was defined above (Section 4). The implication is that actors j and k occupy non-identical positions in the social structure with respect to *all* actors. Therefore, in our reduced-form image of the social structure we assign j and k to *different* aggregate positions or "blocks".

We now consider the largest sets of actors who may jointly be assigned to the *same* position. From Figure 2 it is straightforward to define these sets: families $\langle 1, 2, 13, 15\rangle$, whom we will label "block 1";

families ⟨3, 4, 5, 7, 8, 9, 10, 11, 14, 16⟩, whom we will label "block 2"; the singleton family 6 ("block 3"), and the singleton family 12 ("block 4"). The resemblance of this "relational inclusion" analysis (Figure 2) to the multidimensional scaling of distances between ego algebras (Table 4) is quite close!

A "blockmodel" for a set of relational matrices such as the two shown in Table 1 consists of a partition of the social actors into sets termed "blocks", and a simultaneous representation of each data network by a reduced-form image network. Each node is an image network represents a "block" of individuals. The key feature of the reduced-form representation is that, for any image nodes b_i and b_j which represent sets of social actors B_i and B_j, respectively, *absence* of a tie from b_i to b_j in the image network implies the absence of *all* ties from the set of people B_i to the set of people B_j in the corresponding data network. [14]

The blockmodel obtained by applying the above assignment of families to "blocks", with respect to the $\boldsymbol{L}$ and $\boldsymbol{M}$ matrices of Table 1, is reported as the first two 4×4 matrices in Figure 3 (top panel), which also shows all the distinct compounds (products of any length) formed from the blockmodel image matrices. The lower panel of Figure 3 reports the blockmodel semigroup multiplication table.

What has our particular route to this graph homomorphism (in Sections 4 and 5) purchased for us? All the results we will present in answer to this question arise from comparisons of the data networks (including their semigroup) and the blockmodel image matrices (including their semigroup).

First, we may construct the "cumulated block hierarchy" for the *blocked* data (Figure 3a) in just the same way as we constructed the "cumulated person hierarchy" for the original data (Figure 2). This cumulated block hierarchy is reported in Figure 4 (top panel). Comparison of Figure 4 with Figure 2 yields:

Result 1: The cumulated block hierarchy is a graph homomorphism of the cumulated person hierarchy.

[14] Extensive discussion of blockmodels appears in White *et al.* (1976), Arabie *et al.* (1978), and Breiger (1981). For formal mathematical definitions, see Arabie *et al.* (1978: 31–32), Kim and Roush (1980: 239–252), or White and Reitz (1983: 195). As defined in the text, our concern in this paper is with 0-blockmodels as specified in Arabie *et al.* (1978: 32, Definition 2).

....	XXX.	....	....	.XX.	XXX.	....	XX..	.XX.	XXX.	XX..
.XX.	XX..	.XX.	XX..	.XX.	XXX.	XXX.	XX..	.XX.	XXX.	XX..
.X..	X...	.XX.	XX..	....	XXX.	XXX.	....	.XX.	....	XX..
....	....	....	....	....	....	....	....	....	....	....
L	*M*	*LL*	*LM*	*ML*	*MM*	*LMM*	*MLM*	*MML*	*MLMM*	*MMLM*
1	2	3	4	5	6	7	8	9	10	11

(a) Blockmodel (*L* and *M*) and compound semigroup elements (with blocks as defined from Figure 2).

	1	2	3	4	5	6	7	8	9	10	11
1 = *L*	3	4	3	4	3	7	7	4	3	7	4
2 = *M*	5	6	5	8	9	6	10	11	9	6	11
3 = *LL*	3	4	3	4	3	7	7	4	3	7	4
4 = *LM*	3	7	3	4	3	7	7	4	3	7	4
5 = *ML*	5	8	5	8	5	10	10	8	5	10	8
6 = *MM*	9	6	9	11	9	6	6	11	9	6	11
7 = *LMM*	3	7	3	4	3	7	7	4	3	7	4
8 = *MLM*	5	10	5	8	5	10	10	8	5	10	8
9 = *MML*	9	11	9	11	9	6	6	11	9	6	11
10 = *MLMM*	5	10	5	8	5	10	10	8	5	10	8
11 = *MMLM*	9	6	9	11	9	6	6	11	9	6	11

(b) Semigroup multiplication table for the blocked data.

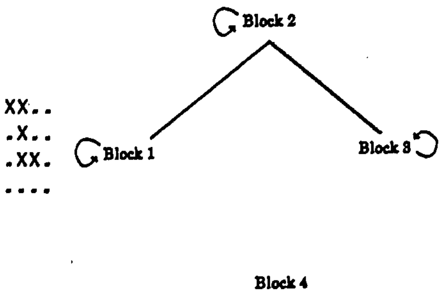
Figure 3. Blockmodel and semigroup multiplication table for Italian data.

That is to say, all relational inclusions (and, therefore, all relational equivalences) at the *individual* level (Figure 2) are maintained at the block level. Relational inclusion among individuals is preserved by the block structure.

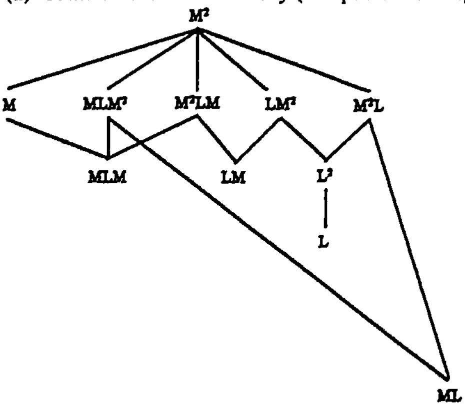
Second, we may examine whether or not the semigroup for the *blocked* data (Figure 3b; a multiplication table of size 11×11) is an algebraic homomorphism of the semigroup generated by the *original* data (Table 2; a multiplication table of size 81×81). This gives us

Result 2. The blockmodel semigroup is precisely a homomorphic image of the data semigroup.

To establish this result, we report in Table 5 the partition of the 81 semigroup elements (of Table 2) which yields the blockmodel semigroup (Figure 3b) as a homomorphic image.



(a) Cumulated block hierarchy (computed from figure 3a).



(b) Inclusion ordering of the semigroup elements in Figure 3a.

Figure 4. Cumulated block hierarchy and inclusion ordering of semigroup elements.

Table 5
Partition of the 81 data semigroup elements (Table 2) into 11 classes that renders the blockmodel semigroup (Figure 3b) a full homomorphism of the data semigroup

Blockmodel class from Figure 3B	Semigroup elements from Table 2
[1] = L	1
[2] = M	2
[3] = L^2	3, 7, 9, 15, 17, 19, 21, 31, 36
[4] = LM	4, 8, 16, 20, 32, 35, 38, 55, 57
[5] = ML	5, 11, 23, 25, 40, 42, 45, 59, 63
[6] = M^2	6, 14, 30, 50, 54, 68, 70, 73, 81
[7] = LM^2	10, 18, 22, 33, 34, 37, 39, 56, 58
[8] = MLM	12, 24, 41, 44, 60, 75, 77
[9] = M^2L	13, 27, 29, 47, 49, 51, 53, 66, 72
[10] = MLM^2	26, 43, 46, 61, 62, 64, 65, 76, 78
[11] = M^2LM	28, 48, 52, 67, 69, 71, 74, 79, 80

Third, we may consider the inclusion order of the 11 blockmodel semigroup elements (panel B of Figure 4) and investigate whether the corresponding inclusion order of the 81 data semigroup elements maps into it. One reason for interest in this question is that the inclusion order of the 81 data semigroup elements is given by the intersection of everyone's W matrices, which are dual to the P matrices. (Recall that examples of W and P matrices, those for Ridolfi, were given in Table 3.) [15] This comparison gives us

Result 3. The inclusion partial order of blockmodel semigroup elements (Figure 4b) is a graph homomorphism of the inclusion order of data semigroup elements.

That is to say, for any of the 81 relations R_1 and R_2 in the semigroup, "R_1 contains R_2" implies that the *class* to which R_1 belongs (see Table 5) contains the *class* to which R_2 belongs (as reported in panel B of Figure 4, which is the inclusion partial order of the blockmodel semigroup elements *and also* a graph homomorphism of the inclusion partial order of data semigroup elements).

Results 1 and 3 are dual to one another. Result 1 (the cumulated block hierarchy is a graph homomorphism of the cumulated person hierarchy) pertains to the individuals' P matrices. Result 3 (the inclusion ordering among blockmodel semigroup elements is a graph homomorphism of the inclusion ordering among data semigroup matrices) pertains to individuals' W matrices, which are dual to P.

[15] Mandel (1978) has shown that the intersection of all individuals' W matrices yields the inclusion partial order of data semigroup elements. How can the analyst compute the intersection of all W matrices, given that the W matrices of each pair of individuals are in general of different sizes? The key is to realize that, in the aggregate, these W matrices index all 81 elements of the semigroup. To illustrate how conformability is obtained, consider again the W matrix of Ridolfi, illustrated in Figure 1. Row i and column i of W both index a class C_i of the 81 semigroup elements, and the set of all 11 such classes induces a partition ("right semigroup homomorphism") of these 81 elements. Form an 81×81 matrix W^* for Ridolfi, in which each row and the corresponding column indexes an element of the data semigroup. Now if entry (i, j) of W is a "1" (reporting that relation i is included in relation j with respect to Ridolfi), enter $n_i \times n_j$ entries of "1" in the subtable of W^* corresponding to the members of class C_i crossed with the members of class C_j (where n_k is the size of class k), indicating that *each* member of class C_i is included within *each* member of class C_j with respect to Ridolfi. The sole purpose of the W^* matrices is to make conformable the W matrices of any *two* individuals. One may now consider the *intersection*, across *all* individuals, of their W^* matrices. Mandel's result states that this intersection is identical to the partial order of inclusions among the 81 16×16 matrices which comprise the elements of the data semigroup (Table 2 above).

Last but not least, we want to investigate the plausibility of our blocking with reference to the attributes of block members. For this purpose John Padgett has kindly provided us with a portion of his data, shown in Table 6, obtained from his combing of the historical materials. We focus in particular on two variables pertaining to the Priorate (*Signoria*), which was, loosely speaking, the municipal council of gentlemen chosen by lot every few months from bags containing the names of those eligible for office. The definition of "eligibility", of

Table 6
Attributes of block members, by block (data of John Padgett) [a]

	(1) 1427 family wealth	(2) Date of first priorate	(3) Number of priors 1282–1344	(4) Combined number of *M* and *L* Ties	(5) Ties of tied
1. Acciaiuoli	10,448	1282	53	2	36.0
2. Albizzi	35,730	1282	65	3	21.3
13. Ridolfi	26,806	1287	38	4	23.0
15. Strozzi	145,896	1283	74	29	8.0
3. Barbadori	55,351	1295	?	14	16.5
4. Bischeri	44,378	1309	12	9	18.3
5. Castellani	19,691	1326	22	18	11.7
7. Guadagni	8,127	1289	21	14	9.8
8. Lamberteschi	41,727	?	0	14	11.8
9. Medici	103,140	1291	53	54	5.5
10. Pazzi	48,233	[b]	[b]	7	16.0
11. Peruzzi	49,313	1283	42	32	10.5
14. Salviati	9,899	1297	35	5	19.5
16. Tornabuoni	48,258	1445	?	7	14.7
6. Ginori	32,013	1344	?	9	13.8
12. Pucci	2,970	?	0	1	8.0

[a] Brief comments on variables ("?" indicates "not available"):

(1) "Net" wealth (reported assets minus deductions, including business loans) from 1427 catasto; to be interpreted with caution.
(2) A public "genealogy"; see text.
(3) A reflection of family influence; see text.
(4) Combined number of business and marriage ties, across the one hundred and sixteen families in Padgett's data set.
(5) Average number of business and marriage ties of the families to whom each family is connected, across all 116 families.

[b] The Pazzi family held the legal designation of "magnate" and thus its members were ineligible to serve as priors. The esteem granted to magnates was not usually matched by their degree of influence.

course, is a key to understanding the social structure of Fifteenth-century Florence; on this point we content ourselves with echoing the observations of Kent (1978: 114–115):

> The idea of distinguished birth came to be ... associated with a family tradition of service in political office.... Suspicious of the value of genealogies flaunted by many of their fellow citizens, contemporaries came in practice to measure the social distinction of a family essentially from the appearance of the first of its members in the Priorate, which became the chief executive magistracy at the time when the republic broadly assumed what was to be its definitive shape and organization for the succeeding 250 years. The *Prioriste*, lists of all those who had held the city's highest office since its inception in 1282, were civic genealogies kept and consulted constantly, not only by government officials, but also by many socially conscious private citizens as a guide to the precise credentials of prospective marriage or business partners.

Table 6 reports the date at which each family was first represented in the Priorate, and the number of priors the family produced from 1282 to 1344. The final two columns of Table 6 refer to tie volume, without distinguishing between marriage and business ties, across all 116 families in Padgett's data set. The final column reports the average number of ties held by those tied to each ego (across all 116 families).

Pucci (block 4), evidently the poorest family, [16] is completely peripheral both in the blockmodel and in the original data matrices. Ginori (block 3), a hanger-on to block 2 in business and a hanger-on to block 1 in marriage (as seen from the blockmodel of Fig. 3), like Pucci, produced no priors by 1344: in fact, both families are representative of the "new men" of relatively recent wealth. With the exception of Peruzzi, *each* family in block 2 first arrived in the priorate later than *any* family in block 1. With the exception of Peruzzi and Medici, all families in block 1 had produced more priors than any family in block 2. With the exception of Strozzi, each family in block 1 is connected to a smaller number of individuals than any member of block, 2 but – again excepting Strozzi – the families to whom block 1 is tied are

[16] These data, taken from the tax records of 1427, are "official" government estimates of net wealth, that is, reported assets minus a complex series of "deductions" including business loans. We interpret these data with corresponding caution.

themselves much more heavily connected than is the case for any member of block 2. Moreover, from the blockmodel of Figure 3 we see that block 1 differs from block 2 by having no *business* ties internally or with the other blocks (although the block 1 members *do* have marriage ties with one another and to other blocks). More qualitatively, block 2 includes those whose major fractional affiliations (either pro- or anti-Medici) crystallized early; block 1 includes families of great influence whose support for one or another of the factions was not forthcoming until late in the struggle. Over all, and with the arguable exceptions we have noted, the blocks as ordered in Table 6 seem to reflect general social standing, with the oldest and most hoary families (block 1) and those with relatively "new" wealth and low participation (blocks 3 and 4) distinguished from the main fractional players (block 2).

Our fourth result, then, is that the blocking identified from the cumulated person hierarchy (Figure 2) seems not unreasonable with respect to the available data on the social identities of these families.

7. Results on other data sets

We have applied our "cumulated person hierarchy" approach to two other data sets, and we very briefly review those applications here. Both data sets are drawn from the classic work-group study of Roethlisberger and Dickson (1939).

We first examined two affect networks: Like and Dislike. Each matrix is entirely symmetric. These networks generate a 43-element semigroup. The cumulated person hierarchy for this application (the construction analogous to Figure 2 above) led us straightaway to a meaningful partition of actors [17] and to the following blockmodel (with blocks listed as defined in the above note):

110000	000110
111000	000000
010000	000110
000000	101101
000000	101000
000000	000100

[17] The sts of equivalent actors were found to be: $A = \langle S1, W4, W7, W8, W9, S4\rangle$, $B = \langle W1, W3\rangle$, $C = \langle I1\rangle$, $D = \langle W5, W6, I3\rangle$, $E = \langle W2\rangle$, and $F = \langle S2\rangle$, with A containing B, C, and F, C containing F, and D containing E and F.

The semigroup generated by this blockmodel is of size 30, and an exact homomorphic image of the 43-element data semigroup.

Our second application to the Roethlisberger–Dickson data concerns the relations Like and Help. Our particular interest here is that the data on Help are emphatically *non*-symmetric. (Of the 91 dyads, 69 are null, 20 are asymmetric, and only 2 are "mutual".) The Like and Help matrices generate a data semigroup of size 93.

As a result of the non-symmetry of the data, we computed each ego's RELE and also his CELE, which are analogous to the RELE but defined on columns. For each ego, we then computed $\boldsymbol{P}^A$ from the RELE (this is new notation for the matrix we have previously labeled $\boldsymbol{P}$) and the analogously-defined $\boldsymbol{P}^B$ from the CELE. We now define $\boldsymbol{P}$ as the intersection of the $\boldsymbol{P}^A$ and $\boldsymbol{P}^B$ of a particular ego. The "cumulated person hierarchy" for this application to *non*-symmetric data was then computed, in accord with our Section 5 discussion, as the union of the $\boldsymbol{P}$ matrices of all individuals taken in turn as ego. This procedure resulted in very little aggregation – perhaps "too little" – at the graph level (ten blocks), while reducing the 93-element data semigroup to a semigroup of size 30. The ten-block partition that we identified is precisely a finer version of the partition found by Winship and Mandel (1983: 334) in their analysis of the same data. [18] Of greatest concern to us was the question of whether the 30-element blockmodel semigroup is an exact semigroup homomorphism of the 93-element data algebra. Once again the answer is yes, precisely so.

8. Generalizations and extensions

How general is our procedure? In this section we discuss this question more formally, and we introduce some additional constructions pertaining to the cumulated person hierarchy. We also present some closely related strategies for data analysis which we believe may prove to be more practical for the non-symmetric case.

[18] Our partition is ⟨W4, W8⟩, ⟨W3⟩, ⟨W9⟩, ⟨W7⟩, ⟨W1⟩, ⟨S4⟩, ⟨S1⟩, ⟨W5, S2, I3⟩, ⟨I1⟩, ⟨W2, W6⟩. The partition of Winship and Mandel (1983: 334) is ⟨W4, W8⟩, ⟨W3, W9, W7⟩, ⟨W1, S4, S1⟩, ⟨W5, S2, I3, I1⟩, ⟨W2, W6⟩.

8.1. A conjecture

We begin with some definitions. Consider the RELE and CELE (the row-elements and the column-elements for a particular ego) defined previously. Represent the matrix of ego's RELE as $\boldsymbol{A}_i$, the entries of which are given by

$$\boldsymbol{A}_i(j, k) = \begin{cases} 1 & \text{if ego } i \text{ has a relation of type } j \text{ to person } k, \\ 0 & \text{otherwise.} \end{cases}$$

The CELE matrix is represented as $\boldsymbol{B}_i$ with entries

$$\boldsymbol{B}_i(j, k) = \begin{cases} 1 & \text{if person } k \text{ has a relation of type } j \text{ to ego } i, \\ 0 & \text{otherwise.} \end{cases}$$

The person hierarchy defined by the RELE is

$$\boldsymbol{P}_i^A(k, m) = \begin{cases} 1 & \text{iff } \boldsymbol{A}_i(j, k) \leqslant \boldsymbol{A}_i(j, m) \text{ for all } j \text{ and there} \\ & \text{is at least one } j \text{ for which } \boldsymbol{A}_i(j, k) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that defined by the CELE is

$$\boldsymbol{P}_i^B(k, m) = \begin{cases} 1 & \text{iff } \boldsymbol{B}_i(j, k) \leqslant \boldsymbol{B}_i(j, m) \text{ for all } j \text{ and there} \\ & \text{is at least one } j \text{ for which } \boldsymbol{B}_i(j, k) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If the data are symmetric, as in the Padgett data set, then $\boldsymbol{P}_i = \boldsymbol{P}_i^A = \boldsymbol{P}_i^B$, where $\boldsymbol{P}_i$ is the person hierarchy matrix defined previously (e.g. Figure 1b). If the data are not symmetric, then we may define $\boldsymbol{P}_i$ as the intersection of the person hierarchy matrices corresponding to the row and column ego algebras. In either case, the *cumulated person hierarchy* matrix is given by $\boldsymbol{U} = \bigcup_i \boldsymbol{P}_i$.

Now we would like to know whether the cumulated person hierarchy is *guaranteed* to lead to a graph homomorphism that is also a semigroup homomorphism. (The assurance of such a guarantee would be important, if only because many of the data semigroups actually encountered in the social networks literature are much larger than those we have analyzed.) We cannot yet provide a definitive answer to this question, but our empirical results suggest a formal conjecture: that the cumulated person hierarchy leads to a network reduction which in turn induces a semigroup reduction. We begin by noting that the manner of construction of the cumulated person hierarchy takes account of ego's relations across all elements of the full data semigroup (unlike the related approaches of Mandel 1983 and Winship and Mandel 1983; footnote 6 above). In addition, we have attempted to locate or construct a counter-example, and have been unable to do so. Attempts to find a proof of the conjecture, however, have so far been unsuccessful.

At a minimum, the "cumulated person hierarchy" construction has been demonstrated to provide a useful *empirical* procedure for dual reduction (of networks of connection *and* their algebras) in a variety of real-data applications.

With respect to formal considerations, we have established several properties of the constructions introduced in this article. For example, it can readily be demonstrated that Result 1 (Section 6) holds for all possible networks, and also that if Result 2 holds for a given network, then so must Result 3. Thus, there is a formal basis for the capturing of properties of the full data network in the reduced-form cumulated person hierarchy. It can further be demonstrated that some other constructions based on the collection of person hierarchies, apart from that of the cumulated person hierarchy, give rise to network reductions inducing semigroup homomorphisms. One of these is described below.

8.2. The central representatives condition

Define the matrix $\boldsymbol{I}$ as the intersection of the matrices $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_n$. Then if $\boldsymbol{I}(m, k) = 1$, k and m can be combined into a single block in which k is a *central representative* for m (Pattison 1982: 93). Specifically, person k is a *central representative* for person m if, for any other person i and for any relation R^*,

iR^*m implies iR^*k *and* mR^*i implies kR^*i.

It follows from Pattison's theorem (1982: 93) that this blocking gives rise to a homomorphism of the network semigroup, as does any blocking of persons $(k, m_1, m_2, \ldots, m_h)$ for which $\boldsymbol{I}(m_1, k) = 1 = \boldsymbol{I}(m_2, k) = \ldots = \boldsymbol{I}(m_h, k)$. Indeed, maximal blockings whose blocks are defined in this way may be constructed from the I matrix. For the data of Table 1, I gives rise to the following five maximal blockings satisfying the Central Representatives Condition: (1, 2); (1, 3); (1, 13); (1, 14); and (1, 16); where it is understood that each family not mentioned in any maximal blocking is assigned to a block by itself. Family 1 (Acciaiuoli) has a marriage tie to family 9 (Medici) and to no other family, while families 2, 3, 13, 14, and 16 have marriage ties to Medici *as well as* other marriage and business ties to other families. Thus, each of the latter families may act as a "central representative" for family 1, as the I matrix records. In fact, in the language of White *et al.* (1976: Appendix A), family 1 is a "floater" with respect to families 2, 3, 13, 14, and 16.

Two features of the blockings induced by I may be observed. First, blockings based on I are likely to be relatively "fine" partitions (that is, reflect less aggregation) in comparison to those based on U. Although the Central Representatives Condition is a generalization of structural equivalence, [19] it is nonetheless a strong condition to impose on network data. Second, not all of the blockings induced by I are strictly finer than the one induced by U, although clearly it is the case that $I(k, m) = 1$ implies $U(k, m) = 1$. Any entry of 1 in the I matrix gives rise to a blocking satisfying the Central Representatives Condition, while persons are blocked by U only if they have identical rows and columns in that matrix.

It is more than likely that we will be able to find some other conditions under which a homomorphism of the data semigroup is guaranteed. In particular, some generalizations of the Central Representatives Condition, possibly along the lines offered by Kim and Roush (1984), seem promising candidates for more general conditions which can be identified from the P_i^A and P_i^B matrices. This is a worthwhile subject for further work.

8.3. Strong components of the network

A final formal consideration is the claim that the procedure for blocking based on the cumulated person hierarchy applies to *strong components* of the network. We may observe that

[19] Note that $I(k, m) = 1 = I(m, k)$ if and only if m and k are structurally equivalent (Lorrain and White 1971).

(i) $U(k, k) = 0$ if $A_i(j, k) = 0$ for all i, j or if $B_i(j, k) = 0$ for all i, j; that is, if k either sends no ties or receives no ties;
(ii) $U(k, k) = 1$ provided that k receives and sends at least one tie;
(iii) $U(k, m) = 1$ only if there exists at least one person with ties to and from k and m.

These observations demonstrate that the "cumulated person hierarchy" construction places two persons in the same block only if they belong to the same strong component of the network (that is, to a subset of persons in the network who are mutually reachable through direct or indirect ties) *or* if *neither* person belongs to a strong component of the network. Indeed, all persons not contained in any strong component will have zero rows and columns in U, and so will be assigned to the same block of the induced blockmodel. These properties of the procedure make it clear that an analysis of the structure *within* each strong component of the network is the analytic focus, and we may note in passing that the analysis of a strong component is unaffected by the addition or by the structure of other strong components. [20] This feature re-iterates the point that the analysis has a motivation which differs sharply from those procedures which seek to identify individuals who have similar *patterns* of relations (Mandel 1983; White and Reitz 1983; Winship and Mandel 1983; Pattison 1986). These authors characterized pattern abstractly, without reference to the particular alters to whom the concrete relations of each ego are expressed. Here, however, we seek a more concrete representation of local roles, one which generalizes structural equivalence *with* reference to actual network connections and yet which maintains an algebraic consistency.

8.4. Data-analytic strategies

For symmetric data such as those we have emphasized in this paper, the cumulated person hierarchy has proved to be a construction useful for empirical analysis. More generally, for non-symmetric data we are

[20] It is also worth noting that the semigroup generated by any collection of strong components of a network is a homomorphic image of the semigroup of the entire network (Pattison 1982).

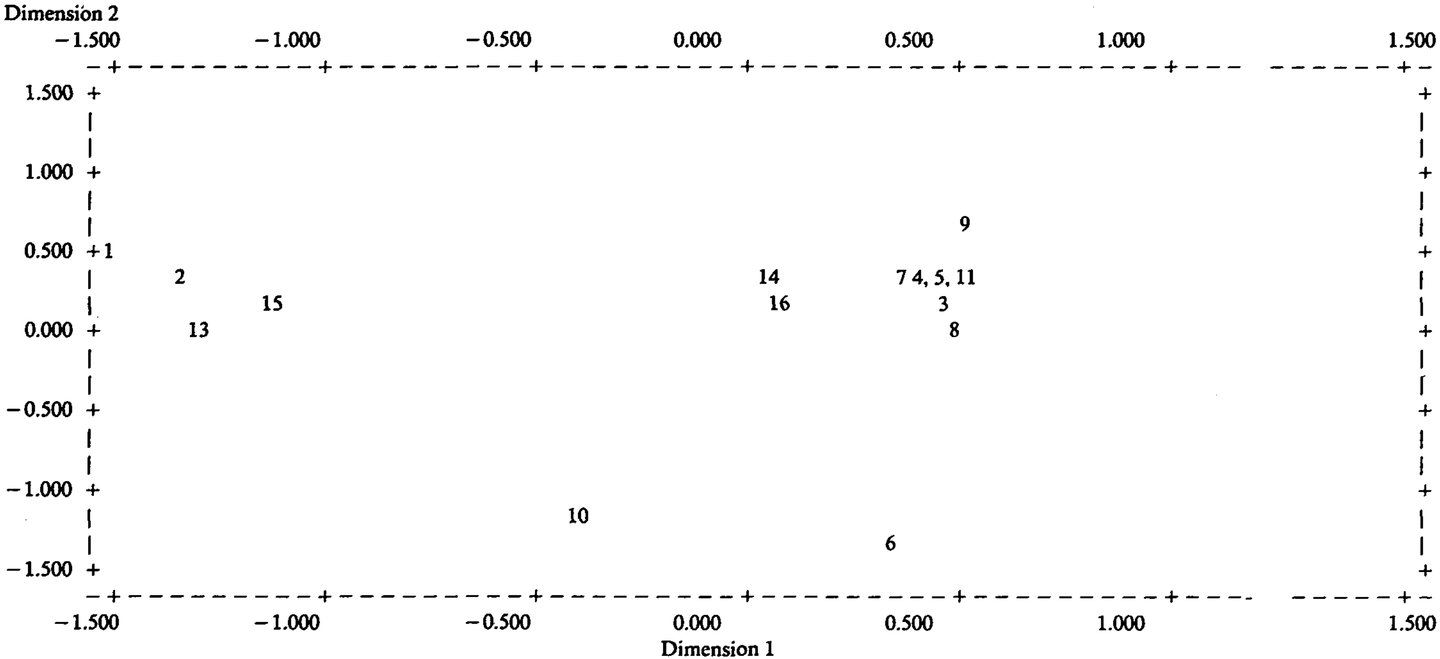
concerned that the generalization just presented (involving the *intersection*, for each ego, of P_i^A and P_i^B), which manifests desirable formal properties, may nonetheless lead to "too little" aggregation of actors. Other strategies for aggregation are possible. While these other strategies may not necessarily induce a homomorphism of the data semigroup, they may nonetheless turn out to provide useful approximations of a semigroup homomorphism in real-data applications. One such strategy involves the aggregation of individuals according to the distance between their ego algebras (left or right semigroup multiplication tables; Section 3 above). Another follows from the construction of a cumulated person hierarchy from only the right (P_i^A) *or* the left (P_i^B) person hierarchies. (It has been suggested elsewhere that separate analyses of ties sent and ties received may yield useful insights for non-symmetric network data; Faust and Romney 1985). A third possible strategy is to begin with some "small" amount of aggregation of the network actors prior to the construction of U as outlined above (see also Pattison 1981 for a related discussion). This alternative seems particularly promising when random errors or systematic bias in the data are suspected (e.g. see Holland and Leinhardt 1973, 1981). Some appropriate methods of preliminary aggregation include those emphasizing structural equivalence (White *et al.* 1976) and those which identify "structurally weak" ties (Schwartz and Sprinzen 1984). Approaches emphasizing a more abstract version of role equivalence (e.g. White and Reitz 1982, 1983; Winship and Mandel 1983; Mandel 1983) seem less appropriate for this particular task, precisely because of their extensions beyond the realm of concrete ties.

9. Comparison with other approaches for the analysis of local structure

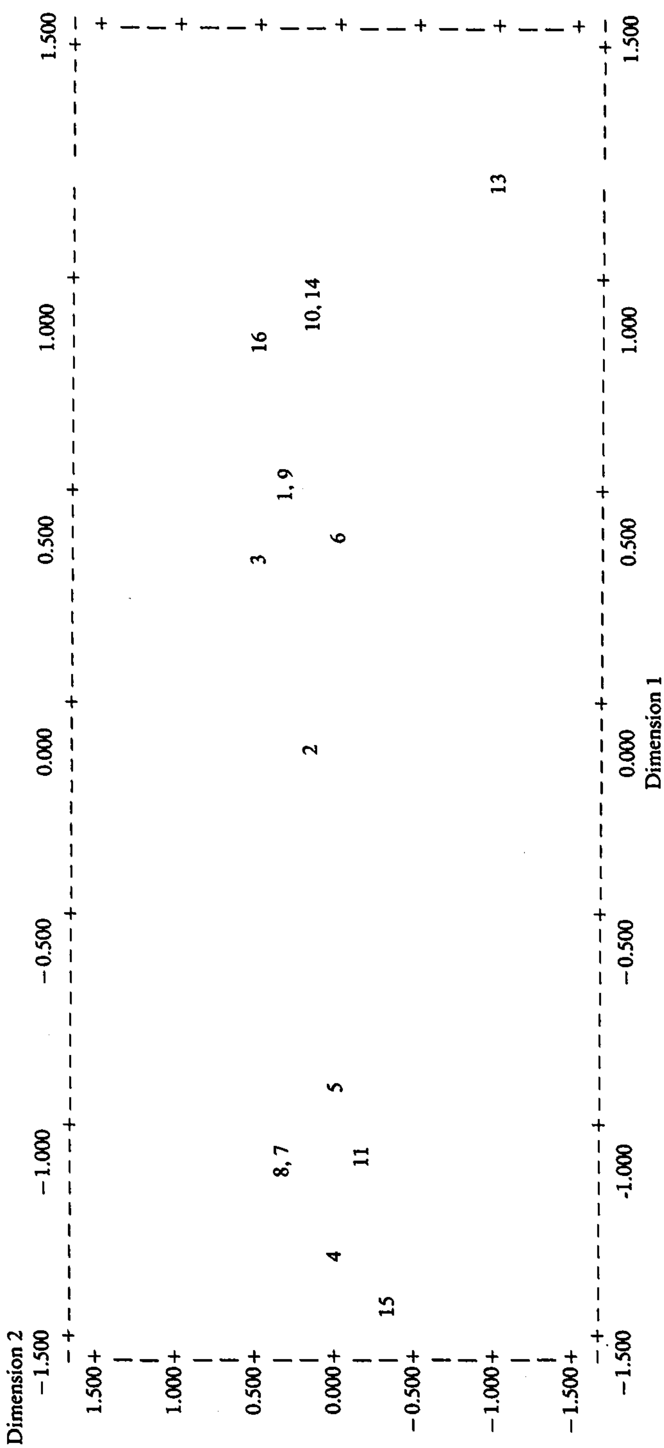
In comparing our procedure with those of "local role equivalence" (Winship and Mandel 1983) and "regular equivalence" (White and Reitz 1982), we first note that *all* these procedures, including the ones introduced in this paper, have as their goal the study of individuals' roles *independently* of any *a priori* partition of the network into positions or blocks. As we noted in the previous section, however, these other procedures have a more abstract emphasis, so that it is useful to compare the results of our analysis with the findings given by these other approaches.

Table 7
Multidimensional scaling of REGE coefficients (panel A) and of coefficients of *P*-distance (panel B) for the Table 1 networks

A. REGE coefficients



B. Coefficients of *P*-distance



9.1. Comparison with regular equivalence and with local equivalence

With respect to regular equivalence, we computed iterated coefficients of "regular multiplex distance" (White and Reitz 1982) for the Table 1 networks. [21] We then eliminated family 12 (Pucci), since it was found to be maximally (and equally) dissimilar from all other families. A multidimensional scaling of coefficients among the remaining 15 families (Kruskal's stress formula 1 = .062) is displayed as panel A of Table 7.

Net of the omission of family 12 from Table 7, panel A, the geometry there accords strikingly with the partition we derived from the cumulated person hierarchy (Figure 2) and also with our geometry of role algebras (Table 4, panel B). Indeed, the correlation of our Table 4, panel A, distances with the REGE coefficients of the "degree" of regular equivalence (omitting family 12, which increases the magnitude of the correlation) is -0.652, which differs significantly from zero and indicates a moderate to strong degree of resemblance. [22] We may interpret this finding as suggesting that for these Florentine families the concrete positional similarities identified by our procedure accord with the more abstract ones sought by White and Reitz (1982). For this strongly interconnected set of families, it is impossible to distinguish the concrete role structure identified by the cumulated person hierarchy from the abstract role structure found by REGE. It is reassuring, though, that the overlap is so strong for these data, and also that we have demonstrated that the analysis has induced a homomorphism of the data semigroup.

To compare our results with those of "local role equivalence", we computed the pair-wise "distance" measure of Winship and Mandel (1983: 329 and Appendix A) for our L and M matrices, using all relations of length one and two. We do not report this distance matrix, since it so closely resembles the coefficient matrix obtained for "regu-

[21] For this purpose we used the REGE program, made available to all researchers through UCINET (Freeman 1985) centered at the University of California, Irvine. We "stacked" the L and M matrices of Table 1, and employed three iterations. A point of terminology: the REGE program produces coefficients of *similarity*, normed to the range 0–1, *not* "distances".

[22] The appropriate significance test for this correlation is "Gamma-1", derived from the quadratic assignment approach of Baker and Hubert (1978); see also Faust and Romney (1985). The Z statistic resulting from this test is -5.06. The sign of the correlation is in the "right" direction since panel A of Table 4 reports distances ("dissimilarities"), while REGE yields coefficients of similarity; see the previous note.

lar equivalence". Omitting family 12, the correlation of the Winship–Mandel "local role" distances and the White–Reitz "regular equivalence" similarities is -0.771 ($Z = -7.45$; see previous footnote). [23] Our further comments here are identical to the ones offered in the previous paragraph concerning regular equivalence.

9.2. A further comparison

We now want to suggest one other comparison, based on a new procedure that is dual to the strategy developed in Mandel (1983). Mandel compared the "truncated relation planes" of two individuals by computing a version of the W matrix for each ego (recall our definition of W as ego's "relation hierarchy" in Section 4 above) and then defining a "measure" of the distance between the pair's W matrices. (The exact computational formula is given in Mandel 1983: 385.) On a continuum from more concrete analyses of roles (as in blockmodel analysis; White *et al.* 1976: 768–772) to those which are more abstract (i.e. algebraic analyses that are expressed independently of the particular network connections between named individuals), Mandel's (1983) approach to local roles, with its emphasis on a version of the W matrix, is by far the most abstract (see also Pattison 1986 for further elaboration). We now propose an orientation that is strictly dual to Mandel's. As such, it provides a much more *concrete* orientation to the analysis of local roles. It allows the study of individuals' roles by making accessible to analysis the similarity of the ordering relations of persons (across all relation types R^*) maintained jointly by all pairs of persons, each considered in his or her capacity as ego. Operationally, we compute Mandel's "distance" measure for all pairs of our 16 families – but with respect to their P matrices, rather than W. A multidimensional scaling of these distances is displayed in Table 7, panel B. [24]

It is immediately apparent that these "dual-P" results differ strikingly from those of all the other procedures we have discussed. For example, the correlation of the distances on which panel B of Table 7 is

[23] The correlation of the Winship–Mandel distances and those of our Table 4, panel A, is 0.589 ($Z = 5.72$).

[24] The configuration displayed in Figure 7b (Kruskal's stress formula 1 = 0.042) suggests that a one-dimensional solution may fit these data quite well. However, we have not pursued this possible solution since the two-dimensional solution illustrates well the division of the families into factions.

based and those of panel A of Table 4 (after omitting family 12) is -0.015 ($Z = -0.14$), and the correlations with regular equivalence and with Winship–Mandel local equivalence are of magnitude 0.05 or below (Z scores of magnitude 0.50 or less).

Rather than define panel B of Table 7 as somehow bizarre or aberrant, we are – on the contrary – excited by these results. The six families in the clear cluster found on the left of panel B of Table 7 – families 4, 5, 7, 8, 11, and 15 – along with families 2 and 3 comprise *precisely* the anti-Medici faction identified by Kent (1978) with respect to the members of our sample. The remaining families – largely centered around Medici (family 9) – comprise *precisely* the pro-Medici faction identified by the same source.

The reconciliation of our results in Table 7, panel B, with the two sets of our earlier results (Table 4, panel B, and Figure 2) and with the results of regular and local equivalence (Table 7, panel A) – all except panel B of Table 7 clearly point in the same direction, while panel B of Table 7 is radically different – follows from the central duality on which our approach rests (Section 4), which is made visible, for example, in Figure 1 above. The more abstract approaches to local roles identify sets of actors who have similar relations to alters, even though these others need not be named. (An example given by Winship and Mandel 1983 concerns the quarterbacks on two opposing football teams; they interact with the same "types" of people but, clearly, not with *identical* people of each "type".) The more concrete approach to local roles identifies, for each ego, sets of actors who have similar relations to the same *named* alters.

These two levels of analysis are quite distinct but – as we have shown – they are nonetheless intimately related. In fact, they are dual, in the sense that operations at one level (for example, inducing additional inclusions in the $\boldsymbol{P}$ matrix of Figure 1) imply corresponding operations at the other level (for example, the addition of inclusions in the $\boldsymbol{W}$ matrix of Figure 1, which may lead to a semigroup homomorphism with respect to a particular ego). In this sense, we have exploited the duality of persons and their algebras.

10. Conclusion

To our knowledge this is the first study to identify full homomorphisms on actual data semigroups (as contrasted with algebraic studies of blockmodel semigroups such as panel B of Figure 3 above; usually *these* are the starting point for algebraic analysis; e.g. Pattison and Bartlett 1982; Bonacich 1983). We have found algebraic simplifications ("semigroup homomorphisms") of three semigroups of hefty sizes (81×81, 43×43, and 93×93 tables). In each case, our construction of the "cumulated person hierarchy" (described in Section 5) led us to a blockmodel ("graph homomorphism") that (a) entailed a "meaningful" partition of social actors, and (b) also provided a semigroup homomorphism. Our substantive and applied procedures therefore seem a natural counterpart to the emerging body of mathematical-theoretical work seeking to specify formal conditions that would allow us "to integrate the analysis of graph homomorphisms and the analysis of semigroup homomorphisms of binary relations under composition" (White and Reitz 1983: 222; see also White and Reitz 1982; Pattison 1982, 1985; Pattison and Bartlett 1982).

In this paper we have not attempted to meet the challenge posed by Bonacich (1983) to relate systematically *all* homomorphisms of a given semigroup to relational features of the data that generated the algebra (but see Pattison 1986). [25] To the contrary, we have sought a *particular* form of dual reduction that seems to allow meaningful data analysis to take place on both the graph and the algebra "levels". Other, probably associated, procedures for relating graph and semigroup homomorphisms now need to be developed to complement the mathematical-theoretical search for formal conditions that is already well under way.

Having employed a variety of techniques of local role analysis on the same data set, we have illustrated the different motivations underlying these techniques even though they often yield similar results. There is no "best" local role analysis in principle; rather, these techniques differ in their ability to provide answers to the somewhat distinct questions that an analyst may pose. Of the three techniques that we have introduced in this paper, the motivation underlying distances between ego algebras (Section 3) is similar to that of Mandel (1983) in searching

[25] It may readily be established, however, that only some of the homomorphisms of a semigroup can *possibly* correspond to network homomorphisms. In particular, only those semigroup homomorphisms which preserve the partially ordered structure of the semigroup may be associated with a network reduction. Such homomorphisms are termed m-homomorphisms (for example, McFadden 1967). It may be argued from this result that we should restrict attention to this class of semigroup homomorphisms when seeking correspondences between network and semigroup reductions.

for similarities in an *abstract* characterization of local roles. The cumulated person hierarchy construction (Section 5) is quite different in emphasis in that it generalizes the concept of structural equivalence in a way which searches for similar patterns of relational inclusions involving a *specific* set of named individuals. The "dual-*P*" approach of Section 9.2, on the other hand, also generalizes structural equivalence in a concrete way, but it measures the approximate similarity of individuals by comparing their individual person hierarchies.

All of the techniques are useful, although not always for the same purpose. We believe that these various purposes will eventually be understood to pertain directly to the central duality of persons and their relations that we have presented.

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